JOURNAL OF ALGEBRA 28, 499–507 (1974)

Automorphism Groups of Complex Lie Algebras

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Communicated by D. Buchsbaum

Received September 22, 1972

1. INTRODUCTION

Let G be a complex affine algebraic group. One can associate with G two automorphism groups. One is the group $\mathscr{W}(G)$ of all affine algebraic group automorphisms, here called the rational automorphisms. Another is the group $\mathscr{A}(G)$ of all (complex) analytic automorphisms. The group $\mathscr{A}(G)$ has a natural structure of a complex Lie group, and $\mathscr{W}(G)$ is a closed complex Lie subgroup of $\mathscr{A}(G)$. In the case where G is connected, it is known from [4] that there is a naturally defined closed normal analytic subgroup $\mathscr{V}(G)$ of $\mathscr{A}(G)$ such that $\mathscr{A}(G)$ is the semidirect product $\mathscr{V}(G) \cdot \mathscr{W}(G)$. Moreover, $\mathscr{V}(G)$ is analytically isomorphic with a complex vector group.

Of special interest is the case where G is the group $\mathscr{A}(L)$ of all Lie algebra automorphisms of a finite-dimensional complex Lie algebra L. We shall see here that only for quite special L's can it happen that $\mathscr{A}(L)$ has nonrational analytic automorphisms. The construction of a Lie algebra L for which this actually does happen is presented in Section 5. It is necessarily somewhat elaborate.

In the exceptional case, if $G = \mathscr{A}(L)$, and G_1 denotes the connected component of the neutral element in G, we shall see that $\mathscr{W}(G_1)_1$ is precisely the group of inner automorphisms of G_1, and that $\mathscr{V}(G_1)$ is isomorphic with the identity component of the group of all analytic characters of G_1. If $\mathscr{V}(G_1)$ is trivial, it follows that $\mathscr{A}(G) = \mathscr{W}(G)$.

Finally, we use results from [3] and [4] in the simplest special case in order to show that, for any complex analytic group G with trivial center, the analytic automorphism tower of G coincides with the rational automorphism tower of $\mathscr{A}(G)$ (from the second term onward), as dealt with in [2].

2. Preparatory Results

If f is a function on a group H and x is an element of H then $x \cdot f$ denotes the function on H given by $(x \cdot f)(y) = f(yx)$. The following known elementary fact will be used several times.

Lemma 2.1. *Let H be an affine algebraic group. Suppose that α is a group automorphism of H inducing on the identity component H_1 a rational automorphism α_1. Then α is a rational automorphism of H.*

Proof. Let P denote the algebra of polynomial functions on H. Let $(x_1, \ldots, x_q)$ be a system of representatives in H for the cosets of H_1 in H. Let f be an element of P, and let u be an element of H_1. We have

$$(f \circ \alpha)(ux_i) = (\alpha(x_i) \cdot f)(\alpha_1(u)).$$

The restriction to H_1 of the element $\alpha(x_i) \cdot f$ of P is a polynomial function. By assumption on α, the same is therefore true for the composite $(\alpha(x_i) \cdot f) \circ \alpha_1$. Therefore, this composite is the restriction to H_1 of some element f_i of P. The equation written above shows that $f \circ \alpha$ coincides with $x_i^{-1} \cdot f_i$ on $H_1 x_i$. Now P contains elements $e_1, \ldots, e_q$ such that the restriction of e_i to $H_1 x_i$ is the constant function with value 1, while its restriction to $H_1 x_j$ is 0 for every j other than i (cf., [1, Theorem 4.4]). We have

$$f \circ \alpha = \sum_{i=1}^{q} e_i x_i^{-1} \cdot f_i ,$$

which is an element of P. Thus, we have shown that $P \circ \alpha \subset P$. The same applies to α^{-1}, so that α is indeed a rational automorphism of H.

Lemma 2.2. *Let L be a finite-dimensional Lie algebra over a perfect field. If L has a non-trivial semisimple automorphism commuting with every derivation of L then the center of L is different from* (0), *and L is the direct Lie algebra sum of its center and the commutator ideal $[L, L]$.*

Proof. Let α denote the given automorphism of L. By [1, Proposition 15.6], we have $L = T + L^\alpha$, where L^α denotes the α-fixed part of L, and T is an α-stable central subalgebra, with $T \cap L^\alpha = (0)$ and $[L^\alpha, L^\alpha] = L^\alpha$. Since α is nontrivial, we have $T \neq (0)$. Choose any nonzero element t from T. Since $T \cap L^\alpha = (0)$, we have $\alpha(t) \neq t$. Now suppose that the center of L^α is not (0). Then there is clearly a linear map δ of L into the center of L^α such that $\delta(t) \neq \delta(\alpha(t))$ and $\delta(L^\alpha) = (0)$. Evidently, δ is a derivation of L and does not commute with α, contradicting our assumption. Hence, the center of L^α is (0), so that T is the center of L. Since $[L, L] = [L^\alpha, L^\alpha] = L^\alpha$, this proves Lemma 2.2.

A Lie algebra M is said to be *perfect* if the center of M is (0) and $[M, M] = M$. Note that *the conclusion of* Lemma 2.2 *is equivalent to the statement that L is the direct Lie algebra sum of a nontrivial abelian Lie algebra and a perfect Lie algebra.* It is almost evident that the converse of Lemma 2.2 is true. Moreover, it is easy to see that the group of all automorphisms such as α is either trivial or isomorphic with the multiplicative group of the base field. Indeed, since $[L, L] = L^{\alpha}$, its elements are left fixed by α. Since T is the center of L, it must be stable under α. Finally, from the fact that α commutes with every derivation, it follows that α must act on T as a scalar multiplication.

3. Nonexceptional Groups $\mathscr{A}(L)$

THEOREM 3.1. *Let L be a finite-dimensional complex Lie algebra, and suppose that L is not the direct sum of a nonzero abelian Lie algebra and a nonzero perfect Lie algebra. Then every analytic automorphism of* $\mathscr{A}(L)$ *is actually a rational automorphism.*

Proof. It is known from [4, end] that the conclusion of Theorem 3.1 holds for $\mathscr{A}(L)_1$ whenever the center of $\mathscr{A}(L)_1$ is unipotent. In this case, we obtain the conclusion for $\mathscr{A}(L)$ by applying Lemma 2.1.

Now suppose that the center of $\mathscr{A}(L)_1$ is not unipotent. Then this center contains a nontrivial semisimple element α. Since the Lie algebra of $\mathscr{A}(L)_1$ is the Lie algebra of derivations of L, the automorphism α commutes with every derivation, so that we can apply Lemma 2.2. This tells us that L is the direct sum of a non-zero abelian Lie algebra and a perfect Lie algebra. The assumption of Theorem 3.1 now leaves us with the case where L is abelian.

In this case, $\mathscr{A}(L)$ is the group of all linear automorphisms of L. In particular, $\mathscr{A}(L)$ is connected, and its unipotent radical is trivial. By [4, Corollary 2], this implies the conclusion of Theorem 3.1.

4. The Exceptional Case

Let L be a finite-dimensional complex Lie algebra not satisfying the assumption of Theorem 3.1. Thus, L is a direct sum $T + M$, where T is abelian and $\neq(0)$, and M is perfect and $\neq(0)$. Then T is the center of L, and $M = [L, L]$. Hence, both T and M are stable under every Lie algebra automorphism of L, so that $\mathscr{A}(L)$ may be identified with the direct product $\mathscr{A}(T) \times \mathscr{A}(M)$. Let us fix the following notation: $P = \mathscr{A}(T)$, $Q = \mathscr{A}(M)_1$, $G = \mathscr{A}(L)_1$. We identify G with $P \times Q$ in the natural fashion, noting that P is the connected full linear group on T. We shall use the notation $\mathscr{A}$, $\mathscr{W}$, $\mathscr{V}$ for automorphism groups as in the introduction.

If H is a complex Lie group such that H/H_1 is finitely generated then the analytic characters of H, i.e., the analytic group homomorphisms of H into the multiplicative group C° of the nonzero complex numbers, consitute a complex Lie group. We shall denote the connected component of the neutral element in this group by $\mathscr{K}(H)$. In our case, H will be an affine algebraic group. The commutator factor group $H/[H, H]$ will be a direct product of a vector group and a reductive group, the vector group being the unipotent radical. The elements of $\mathscr{K}(H)$ induce analytic homomorphisms $H/[H, H] \to C^\circ$, which are trivial on the reductive factor. On the vector group factor, they are of the form $(c_1, \ldots, c_m) \to \exp(a_1c_1 + \cdots + a_mc_m)$, where the a_i's are fixed complex numbers depending only on the given element of $\mathscr{K}(H)$. In this way, $\mathscr{K}(H)$ is isomorphic with the vector group factor of $H/[H, H]$, i.e., with the unipotent radical $(H/[H, H])_u$.

THEOREM 4.1. *In the above notation,* $\mathscr{W}(G)_1$ *coincides with the group of inner automorphisms of* G. *The nonrational part* $\mathscr{V}(G)$ *of* $\mathscr{A}(G)$ *is isomorphic with* $\mathscr{K}(G)$, *and with* $\mathscr{K}(Q)$, *and with the complex vector group* $(Q/[Q, Q])_u = (G/[G, G])_u$. *Finally* $\mathscr{A}(G)_1 \approx \mathscr{K}(Q) \times \mathscr{W}(G)_1$.

Proof. Since P is the full linear group on T, the center of P is isomorphic with C°. Since the center of M is trivial, so is the center of its derivation algebra $\mathscr{D}(M)$. Now $\mathscr{D}(M)$ is the Lie algebra of Q. Hence, the center of Q is discrete. It follows that the center of P is the connected component of the neutral element in the center of G. Therefore, the center of P is stable under the action of $\mathscr{A}(G)$, and its elements are fixed under the action of $\mathscr{A}(G)_1$.

Next, note that P is the product of its center and its commutator subgroup $[P, P]$. The Lie algebra of $[P, P]$ is an ideal J of the derivation algebra $\mathscr{D}(L)$, and $[J, J] = J$. Evidently, this implies that every derivation of $\mathscr{D}(L)$ stabilizes J. Considering the adjoint representation of $\mathscr{A}(G)$ on $\mathscr{D}(L)$, we see from this that J, and, therefore, also $[P, P]$, is stable under the action of $\mathscr{A}(G)_1$. Thus, we conclude that P is stable under the action of $\mathscr{A}(G)_1$.

Let α be an element of $\mathscr{A}(G)_1$. Since $\alpha(P) = P$, it is clear that $\alpha(Q)$ lies in the centralizer of P. This centralizer is $C^\circ \times Q$, where we have identified the center of P with C°. Therefore, $\alpha(q) = (\alpha_1(q), \alpha_0(Q))$ for every element q of Q, where α_1 is an analytic map $Q \to C^\circ$ and α_0 is an analytic map $Q \to Q$. From the fact that α leaves the elements of C° fixed, we see that α_1 is a group homomorphism, and the same evidently holds for α_0. Moreover, α_0 is invertible, with inverse $(\alpha^{-1})_0$. Clearly, therefore, we have $\alpha_1 \in \mathscr{K}(Q)$ and $\alpha_0 \in \mathscr{A}(Q)_1$. Conversely, if ρ and σ are arbitrary elements of $\mathscr{K}(Q)$ and $\mathscr{A}(Q)_1$, respectively, we can define an element α of $\mathscr{A}(G)_1$ by $\alpha(p, q) = (p\rho(q), \sigma(q))$, so that $\alpha_1 = \rho$ and $\alpha_0 = \sigma$.

Let Y denote the element-wise fixer of P in $\mathscr{A}(G)_1$. Then $\mathscr{A}(G)_1$ may be identified with the direct product $\mathscr{A}(P)_1 \times Y$, and the above shows that Y may be identified with the appropriate semidirect product $\mathscr{K}(Q) \cdot \mathscr{A}(Q)_1$.

For the full linear group P, it is a standard fact that $\mathscr{A}(P)_1$ coincides with the group of inner automorphisms. On the other hand, recall that the Lie algebra of Q is the derivation algebra of the perfect Lie algebra M. It is a well known elementary fact that every derivation of the derivation algebra of a perfect Lie algebra is an inner derivation. By the usual familiar argument, it follows from this that $\mathscr{A}(Q)_1$ coincides with the group of inner automorphisms of Q. This implies that the conjugation action of $\mathscr{A}(Q)_1$ on $\mathscr{K}(Q)$ within Y is trivial.

Assembling the above results, we see that $\mathscr{A}(G)_1$ may be identified with the direct product $\mathscr{K}(Q) \times \mathscr{I}(G)$, where $\mathscr{I}(G)$ denotes the group of inner automorphisms of G. Clearly, $\mathscr{I}(G) \subset \mathscr{W}(G)_1$. As we have seen in the discussion immediately preceding the statement of Theorem 4.1, the automorphisms of G corresponding to the elements of $\mathscr{K}(Q)$ are determined by their effect on the unipotent radical of $Q/[Q, Q]$. This makes it evident that they are not rational, except for the trivial one. It follows that we must have $\mathscr{I}(G) = \mathscr{W}(G)_1$. Moreover, if one looks at the definition of $\mathscr{V}(G)$, as given in the beginning of [4], one sees almost immediately that the automorphisms corresponding to the elements of $\mathscr{K}(Q)$ belong to $\mathscr{V}(G)$. The above now shows that no other elements of $\mathscr{A}(G)_1$ can belong to $\mathscr{V}(G)$. Thus, the subgroup of $\mathscr{A}(G)$ corresponding to $\mathscr{K}(Q)$ must coincide with $\mathscr{V}(G)$.

Finally, since P is reductive, $\mathscr{K}(P)$ is trivial, so that $\mathscr{K}(Q)$ may be identified with $\mathscr{K}(G)$. The proof of Theorem 4.1 is now complete.

It is clear from Lemma 2.1 and Theorems 3.1 and 4.1 that $\mathscr{A}(L)$ *can have nonrational analytic automorphisms only if L is as in* Theorem 4.1 *and $G/[G, G]$ is not reductive.* Any example of such a Lie algebra L is necessarily somewhat complicated. For instance, it is not difficult to see that the radical of L cannot be abelian.

5. An Example

We construct a Lie algebra L such that $\mathscr{A}(L)$ has nonrational analytic automorphisms. Let S denote the three-dimensional simple complex Lie algebra. Let S' denote a copy of S, viewed as an S-module via the adjoint representation. We indicate a selected S-module isomorphism $S \to S'$ by $s \to s'$. Let S'' denote the homogeneous component of degree 2 of the exterior algebra built over S', viewed as an S-module in the natural fashion. The elements of S'' are the sums of exterior products $x'y'$, with x and y ranging over S. The Lie multiplication of S yields an S-module isomorphism $S'' \to S$, sending $x'y'$ onto $[x, y]$. We indicate the inverse of this isomorphism by

$s \to s''$. Now we make the direct S-module sum $S' + S''$ into a Lie algebra such that $[S'', S' + S''] = (0)$ and $[S', S'] = S''$, with $[x', y'] = x'y'$. Evidently, the S-module structure of $S' + S''$ makes S act by Lie algebra derivations.

Let B be a simple nontrivial S-module (to be specified later), and let B' and B'' be isomorphic copies of B. We select S-module isomorphisms $B \to B'$, $B' \to B''$ and $B \to B''$, making the third equal to the composite of the first two. We indicate these in computations by $b \to b'$, $b' \to (b')' = b''$, and $b \to b''$, respectively. There is a natural representation of the Lie algebra $S' + S''$ on the direct sum $B + B' + B''$, as follows. The component B'' is annihilated by $S' + S''$. The component B' is annihilated by S'', and $S' \cdot B' = B''$, with $s' \cdot b' = (s \cdot b)''$. Finally, the action on B is given by $S' \cdot B = B'$, with $s' \cdot b = (s \cdot b)'$, and $S'' \cdot B = B''$, with $s'' \cdot b = (s \cdot b)''$.

Accordingly, we make $S' + S'' + B + B' + B''$ into a Lie algebra, the semidirect sum of the abelian ideal $B + B' + B''$ and the subalgebra $S' + S''$. This makes $[S', B] = B'$, and $[S', B'] = B'' = [S'', B]$, with $[s', b] = s' \cdot b = (s \cdot b)'$, etc., in accordance with the representation defined above. The module action of S is still an action by Lie algebra derivations, as a straightforward verification will show.

Now we adjoin three more simple nontrivial S-modules C, D, E as direct S-module summands, and we extend the Lie multiplication so as to obtain the following.

$$[C, D] = E = [B, D] \qquad \text{and} \qquad [S', C] = B''.$$

All the remaining Lie products of pairs of S-simple components will be defined to be (0). The Lie algebra, thus obtained, will be

$$R = S' + S'' + B + B' + B'' + C + D + E.$$

The center of R will be $B'' + E$. Moreover, we shall have $[R, R] = S'' + B' + B'' + E$. The only component not annihilating $[R, R]$ will be S', and we shall have

$$[R, [R, R]] = [S', [R, R]] = [S', B'] = B''.$$

If we adjoin C first, and then D and E, we see easily that the Jacobi identity continues to hold; all the new triple Lie products will be 0.

It remains to make suitable choices for the modules B, C, D, E, where we wish to ensure also that S will act by Lie algebra derivations on R. Recall that, for every positive integer n, there is, to within isomorphisms, exactly one simple S-module of dimension n, say A_n. Our modules S' and S'' are isomorphic with A_3. We choose

$$B = A_4, \qquad C = A_6, \qquad D = A_2, \qquad E = A_5.$$

The tensor product decompositions we use below are immediately obtained from the well known Clebsch–Gordan formula. First, $A_6 \otimes A_2 \approx A_7 + A_5$. For c in C and d in D, we define $[c, d] = -[d, c]$ as the projection image of $c \otimes d$ in the component $A_5 = E$. Then we have $[C, D] = E$. Similarly, using that $A_4 \otimes A_2 \approx A_5 + A_3$, we define $[B, D] = E$. Finally, we define $[S', C] = B'' \approx A_4$ in the same way, using that $A_3 \otimes A_6 \approx A_8 + A_6 + A_4$. These definitions are easily seen to ensure that S acts by Lie algebra derivations on R.

Accordingly, we form the corresponding semidirect sum Lie algebra $M = R + S$, whose radical is the nilpotent Lie algebra R. Clearly, $[S, R] = R$, whence $[M, M] = M$. Moreover, the centralizer of S in M is (0). A fortiori, the center of M is (0), so that M is a perfect Lie algebra.

Now we examine the automorphism group $\mathscr{A}(M)$. Let us denote the element-wise fixer of S in $\mathscr{A}(M)$ by $\mathscr{A}_S(M)$. Let τ be any element of $\mathscr{A}_S(M)$, and let s', b, c, d be elements of S', B, C, D, respectively. Using only that τ is an S-module automorphism and that R and $[R, R]$ are τ-stable, we see by the usual Schur's Lemma argument that there are nonzero complex numbers $\alpha, \beta, \gamma, \delta$, depending only on τ, such that

$$\begin{aligned} &\tau(s') = \alpha s' + s^*, && \text{with} \quad s^* \in S'', \\ &\tau(b) = \beta b + b^*, && \text{with} \quad b^* \in B' + B'', \\ &\tau(c) = \gamma c && \text{and} \quad \tau(d) = \delta d. \end{aligned}$$

From $[C, D] = E$, we find that $\tau(e) = \gamma\delta e$ for every element e of E. On the other hand, from $[B, D] = E$, we obtain $\tau(e) = \beta\delta e$. Therefore, we must have $\gamma = \beta$. Since $[S', C] = B''$, the effect of τ on B'' is the scalar multiplication by $\alpha\gamma$. On the other hand, we have also $[S', [S', B]] = B''$, whence we see that the effect of τ on B'' is the scalar multiplication by $\alpha^2\beta$. Hence, $\alpha^2\beta = \alpha\gamma = \alpha\beta$, so that $\alpha = 1$. Moreover, the map $S' \to S''$ sending each s' onto $\tau(s') - s' = s^*$ is an S-module homomorphism and, therefore, must be a scalar multiple of the S-module isomorphism sending each s' onto s''. Thus, there is a complex number ρ, depending only on τ, such that $\tau(s') = s' + \rho s''$ for every element s' of S'. Let us observe also that τ leaves the elements of S'' fixed, because $S'' = [S', S']$.

In particular, the above results imply that the commutator subgroup $[\mathscr{A}_S(M), \mathscr{A}_S(M)]$ leaves the elements of S' fixed. On the other hand, given an arbitrary complex number ρ, we can define an element ρ^* of $\mathscr{A}_S(M)$ as follows. For every s' in S', we put $\rho^*(s') = s' + \rho s''$. For every element b' of B', we put $\rho^*(b') = b' + \rho b''$. Finally, the elements of all the remaining S-simple components of M are to be left fixed by ρ^*. The only nontrivial part of the verification that ρ^* is a Lie algebra automorphism is as follows.

$$[\rho^*(s'), \rho^*(b)] = [s' + \rho s'', b] = [s', b] + \rho[s'', b] \qquad \text{and} \qquad [s'', b] = [s', b]'.$$

We wish to prove that if $\rho \neq 0$ then ρ^* does not belong to $[\mathscr{A}(M), \mathscr{A}(M)]$. For this purpose, we have to analyze $\mathscr{A}(M)$ to some extent. If x is any element of M, we denote by D_x the inner derivation effected by x, so that $D_x(y) = [x, y]$. By ordinary exponentiation of linear endomorphisms, every element x of M determines an element $\mathrm{Exp}(D_x)$ of $\mathscr{A}(M)$. The subgroup of $\mathscr{A}(M)$ that is generated by these automorphisms is called the group of inner automorphisms of M, and we shall denote it by $\mathscr{I}(M)$. It is easy to see that $\mathscr{I}(M)$ is normal in $\mathscr{A}(M)$. By a well known result from Lie algebra theory (due to Malcev), every maximal semisimple Lie subalgebra of M is of the form $\mathrm{Exp}(D_r)(S)$, with r in R. It follows that every element of $\mathscr{A}(M)$ is the product of an element of $\mathscr{I}(M)$ and an automorphism stabilizing S. For the simple three-dimensional Lie algebra S, it is well known that $\mathscr{A}(S) = \mathscr{I}(S)$. Since every element of $\mathscr{I}(S)$ is the restriction to S of an element of $\mathscr{I}(M)$, we conclude that $\mathscr{A}(M) = \mathscr{I}(M)\mathscr{A}_S(M)$.

Let J denote the subgroup of $\mathscr{I}(M)$ that is generated by the automorphisms $\mathrm{Exp}(D_s)$ with s in S. It follows from the nilpotency of R and the Campbell–Hausdorff formula that every element of $\mathscr{I}(M)$ is the product of an element of J and an automorphism $\mathrm{Exp}(D_r)$ with r in R. From the fact that the centralizer of S in R is (0), we see that $\mathrm{Exp}(D_r)$ stabilizes S only if $r = 0$. It follows that $\mathscr{I}(M) \cap \mathscr{A}_S(M) \subset J$, and, therefore, stabilizes S'. Clearly,

$$[\mathscr{A}(M), \mathscr{A}(M)] \cap \mathscr{A}_S(M) \subset (\mathscr{I}(M) \cap \mathscr{A}_S(M))[\mathscr{A}_S(M), \mathscr{A}_S(M)]$$

Recalling our earlier result about $\mathscr{A}_S(M)$, we see that

$$[\mathscr{A}(M), \mathscr{A}(M)] \cap \mathscr{A}_S(M)$$

stabilizes S' (actually, it fixes the elements of S'). Therefore, the automorphisms ρ^* defined above do not belong to $[\mathscr{A}(M), \mathscr{A}(M)]$, except when $\rho = 0$. Clearly, these automorphisms ρ^* constitute a one-dimensional complex analytic subgroup V of $\mathscr{A}(M)$ whose action on M is unipotent. Hence, V is actually a unipotent algebraic subgroup of $\mathscr{A}(M)$. Since V is not contained in $[\mathscr{A}(M), \mathscr{A}(M)]$, its canonical image in $\mathscr{A}(M)/[\mathscr{A}(M), \mathscr{A}(M)]$ is a nontrivial unipotent algebraic subgroup. Writing Q for $\mathscr{A}(M)$, we have the result that $(Q/[Q, Q])_u$ is nontrivial, so that $\mathscr{K}(Q)$ is nontrivial.

Let L be the direct Lie algebra sum of M and a one-dimensional Lie algebra. Then, as explained in Section 3, the nontrivial elements of $\mathscr{K}(Q)$ yield nonrational analytic automorphisms of $\mathscr{A}(L)$.

6. Automorphism Towers

Let G be a complex analytic group with trivial center, and let L denote the Lie algebra of G. The kernel of the universal covering of G coincides with the center of the universal covering group. It follows that the natural homomorphism $\mathscr{A}(G) \to \mathscr{A}(L)$ is an isomorphism of complex analytic groups, in the present case. In particular, $\mathscr{A}(G)$ is therefore an affine algebraic group. Moreover, our assumption on G directly implies that the centralizer of $\mathscr{A}(G)_1$ in $\mathscr{A}(G)$ is trivial (consider the inner automorphisms).

Let us write H for $\mathscr{A}(G)$. Then H is a complex affine algebraic group such that the centralizer of H_1 in H is trivial. Since the center of H_1 is trivial, we know from [4, end] that $\mathscr{A}(H_1)$ coincides with $\mathscr{W}(H_1)$ and is algebraic. By Lemma 2.1, the first implies that $\mathscr{A}(H) = \mathscr{W}(H)$. By [3, Props. 2.2, 2.3], the fact that $\mathscr{W}(H_1)$ is algebraic implies that $\mathscr{W}(H)$ is algebraic. Now $\mathscr{A}(H)$ has all the properties we have just used for H: it is a complex affine algebraic group; the centralizer of $\mathscr{A}(H)_1$ in $\mathscr{A}(H)$ is trivial, by [2, Proposition 2.1]. Therefore, we can repeat this argument indefinitely and obtain the following result.

THEOREM 6.1. *For every complex analytic group G with trivial center, the analytic automorphism tower $G \to \mathscr{A}(G) \to \mathscr{A}(\mathscr{A}(G)) \to \cdots$ coincides (from the second term onward) with the algebraic automorphism tower $G \to H \to \mathscr{W}(H) \to \mathscr{W}(\mathscr{W}(H)) \to \cdots$.*

Therefore, the results of [2] are directly applicable to the above analytic situation.

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